# 2.1 The Tangent and Velocity Problems

**1.** A tank holds 1000 gallons of water, which drains from the bottom of the tank in half an hour. The values in the table show the volume $V$ of water remaining in the tank (in gallons) after $t$ minutes.

| $t$ (min) | 5 | 10 | 15 | 20 | 25 | 30 |
|---|---|---|---|---|---|---|
| $V$ (gal) | 694 | 444 | 250 | 111 | 28 | 0 |

(a) If $P$ is the point (15, 250) on the graph of $V$, find the slopes of the secant lines $PQ$ when $Q$ is the point on the graph with $t = 5$, 10, 20, 25, and 30.

(b) Estimate the slope of the tangent line at $P$ by averaging the slopes of two secant lines.

(c) Use a graph of the function to estimate the slope of the tangent line at $P$. (This slope represents the rate at which the water is flowing from the tank after 15 minutes.)

Solution

a) Calculate slope of PQ

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$P = (15, 250)$ $(x_1, y_1)$

| $x_2$ <br> t (min) | $y_2$ <br> V (gal) | $m_{PQ}$ |
|---|---|---|
| 5 | 694 | $\frac{694-250}{5-15} = -44.4$ |

| | | |
|---|---|---|
| 10 | 444 | $\frac{444-250}{10-15} = -38.8$ |
| 20 | 111 | ... $-27.8$ |
| 25 | 28 | ... $-22.2$ |
| 30 | 0 | ... $-16.\bar{6}$ |

b) Using the values of $t$ that correspond to the points closest to $P$ ($t=10$ and $t=20$), we have

$$\frac{-38.8+(-27.8)}{2} = -33.3$$

c) From the graph, we can estimate the slope of the tangent line at $P$ to be

$$\frac{-300}{9} = -33.\bar{3}$$

## 2.2 The Limit of a Function

**1.** Explain in your own words what is meant by the equation

$$\lim_{x \to 2} f(x) = 5$$

Is it possible for this statement to be true and yet $f(2) = 3$? Explain.

Solution

The limit as $x$ approaches 2 of $f(x)$ is 5
or
As $x$ approaches 2, $f(x)$ approaches 5

Yes it is possible. There could be a hole in the graph at (2, 5), and be defined at (2, 3)

**2.** Explain what it means to say that

$$\lim_{x \to 1^-} f(x) = 3 \qquad \text{and} \qquad \lim_{x \to 1^+} f(x) = 7$$

In this situation is it possible that $\lim_{x \to 1} f(x)$ exists? Explain.

Solution

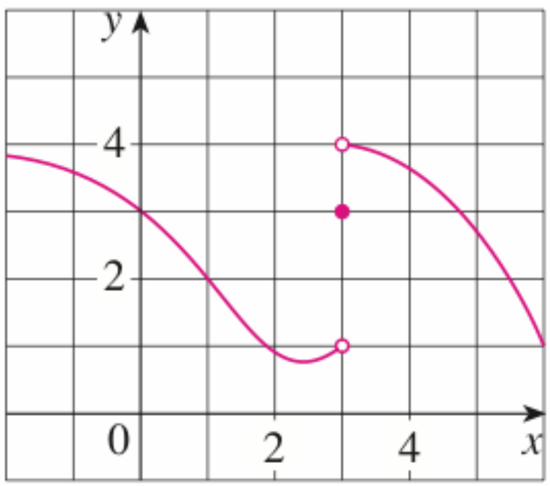
The limit as $x$ approaches 1 from the left of $f(x)$ is 3
and
The limit as $x$ approaches 1 from the right of $f(x)$ is 7

The limit as $x$ approaches 1 does not exist because the limit from the left does not equal the limit from the right

**5.** For the function $f$ whose graph is given, state the value of each quantity, if it exists. If it does not exist, explain why.

(a) $\lim_{x \to 1} f(x)$ (b) $\lim_{x \to 3^-} f(x)$ (c) $\lim_{x \to 3^+} f(x)$

(d) $\lim_{x \to 3} f(x)$ (e) $f(3)$




Solution

a) 2 b) 1 c) 4 d) This limit does not exist because the limit from the left does not equal the limit from the right

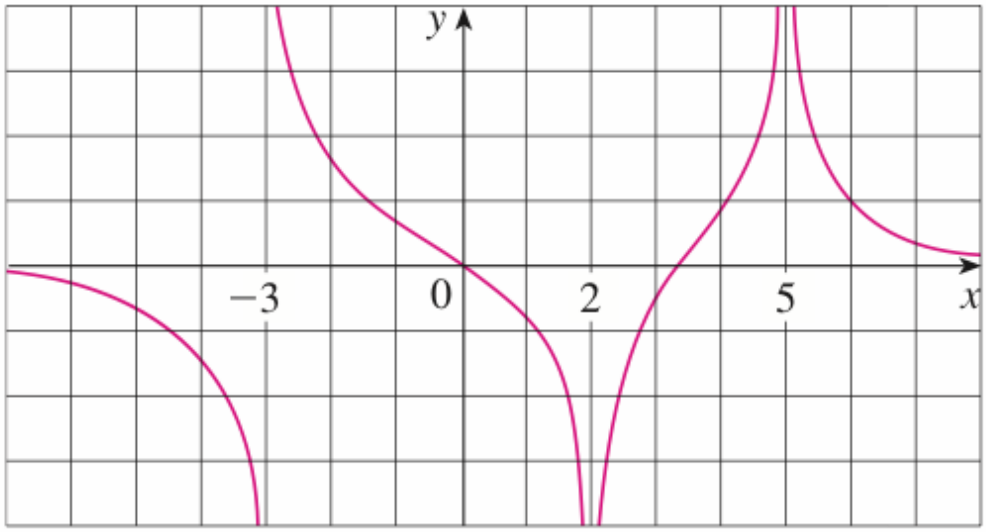
e) 3

**8.** For the function $R$ whose graph is shown, state the following.

(a) $\lim_{x \to 2} R(x)$ (b) $\lim_{x \to 5} R(x)$

(c) $\lim_{x \to -3^-} R(x)$ (d) $\lim_{x \to -3^+} R(x)$

(e) The equations of the vertical asymptotes.




Solution

a) $-\infty$ b) $\infty$ c) $-\infty$ d) $\infty$

e) $x=-3$ $x=2$ $x=5$

Guess the limit using a table of the following values

**20.** $\lim_{x \to -1} \dfrac{x^2 - 2x}{x^2 - x - 2}$,

$x = 0, -0.5, -0.9, -0.95, -0.99, -0.999,$
$-2, -1.5, -1.1, -1.01, -1.001$

Solution

| $x$ | $f(x)$ |
|---|---|
| 0 | 0 |
| -0.5 | -1 |
| -0.9 | -9 |
| -0.95 | -19 |
| -0.99 | -99 |
| -0.999 | -999 |

| $x$ | $f(x)$ |
|---|---|
| -2 | 2 |
| -1.5 | 3 |
| -1.1 | 11 |
| -1.01 | 101 |
| -1.001 | 1001 |

The limit <u>does not exist</u> because $\lim_{x\to -1^-} f(x) = \infty$ and $\lim_{x\to -1^+} f(x) = -\infty$

**22.** $\lim_{h\to 0} \frac{(2+h)^5 - 32}{h}$,
$h = \pm 0.5, \pm 0.1, \pm 0.01, \pm 0.001, \pm 0.0001$

round to 6 decimal places

Solution

| $h$ | $f(h)$ |
|---|---|
| 0.5 | 131.312500 |
| 0.1 | 88.410100 |
| 0.01 | 80.804010 |
| 0.001 | 80.080040 |
| 0.0001 | 80.008000 |

| $h$ | $f(h)$ |
|---|---|
| -0.5 | 48.812500 |
| -0.1 | 72.390100 |
| -0.01 | 79.203990 |
| -0.001 | 79.920040 |
| -0.0001 | 79.992000 |

From these tables, it appears that $\lim_{h\to 0} \frac{(2+h)^5 - 32}{h} = 80$

# 2.3 Calculating Limits using Limit Laws

**1.** Given that

$$\lim_{x \to 2} f(x) = 4 \qquad \lim_{x \to 2} g(x) = -2 \qquad \lim_{x \to 2} h(x) = 0$$

find the limits that exist. If the limit does not exist, explain why.

(a) $\lim_{x \to 2} [f(x) + 5g(x)]$ (b) $\lim_{x \to 2} [g(x)]^3$

(c) $\lim_{x \to 2} \sqrt{f(x)}$ (d) $\lim_{x \to 2} \dfrac{3f(x)}{g(x)}$

(e) $\lim_{x \to 2} \dfrac{g(x)}{h(x)}$ (f) $\lim_{x \to 2} \dfrac{g(x)h(x)}{f(x)}$

## Solution

a) $\lim_{x \to 2} [4 + 5(-2)]$

$= 4 - 10$

$= -6$

b) $\lim_{x \to 2} [g(x)]^3$

$= (-2)^3$

$= -8$

c) $\lim_{x \to 2} \sqrt{4}$

$= 2$

d) $\lim_{x \to 2} \dfrac{3(4)}{-2}$

$= \dfrac{12}{-2}$

$= -6$

e) $\lim \dfrac{-2}{0}$

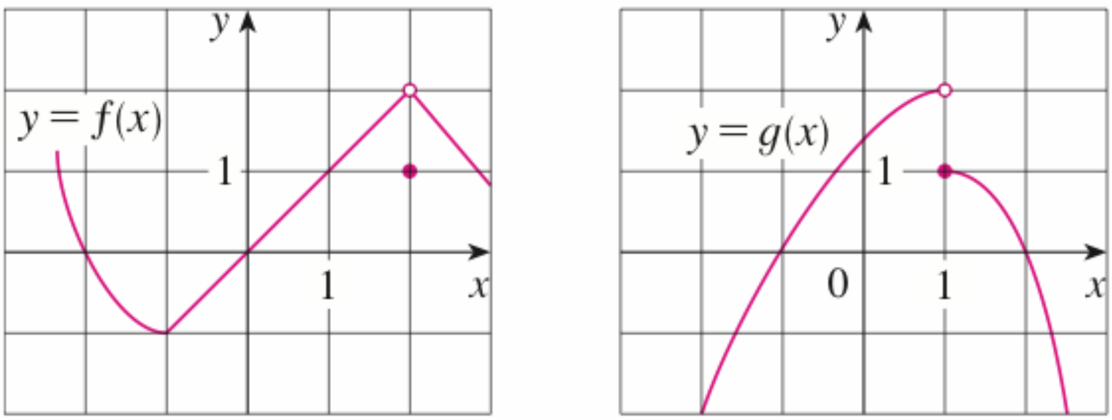
f) $\lim \dfrac{-2(0)}{4}$

$x \to 2$ $\qquad\qquad\qquad\qquad$ $x \to 2$

We cannot evaluate this limit because the denominator is 0 (DNE) $\qquad$ $= \frac{0}{4}$ $\quad = 0$

* infinite limit → $\frac{\#}{0}$ *

**2.** The graphs of $f$ and $g$ are given. Use them to evaluate each limit, if it exists. If the limit does not exist, explain why.




(a) $\lim_{x\to 2} [f(x) + g(x)]$ $\qquad$ (b) $\lim_{x\to 1} [f(x) + g(x)]$

(c) $\lim_{x\to 0} [f(x)g(x)]$ $\qquad$ (d) $\lim_{x\to -1} \dfrac{f(x)}{g(x)}$

(e) $\lim_{x\to 2} [x^3 f(x)]$ $\qquad$ (f) $\lim_{x\to 1} \sqrt{3 + f(x)}$

Solution

a) $\lim_{x\to 2} f(x) = 2$ $\quad$ $\lim_{x\to 2} g(x) = 0$

$(2+0)$
$= 2$

b) $\lim_{x\to 1} f(x) = 1$

$\lim_{x\to 1} g(x) = DNE$ because the limit from the right does not equal the limit from the left

c) $\lim_{x\to 0} f(x) = 0$

$\lim_{x\to 0} g(x) = 4/3$

$(0 \cdot 4/3)$
$= 0$

d) $\lim_{x\to -1} f(x) = -1$

$\lim_{x\to -1} g(x) = 0$

$= \frac{-1}{0}$ This limit does not exist because the denominator is 0

e) $x = 2$

$\lim_{x\to 2} f(x) = 2$

$(2^3)(2)$
$= 16$

f) $\lim_{x\to 1} f(x) = 1$

$\sqrt{3+1}$
$= \sqrt{4}$
$= 2$

Evaluate the following limits

**11.** $\lim_{x \to 5} \dfrac{x^2 - 6x + 5}{x - 5}$

$\dfrac{x^2-6x+5}{x-5}$

if you try to plug in 5, the numerator and denominator will both be 0

$\dfrac{(x-5)(x-1)}{(x-5)}$

* $\frac{0}{0}$ means there is a removable discontinuity (cancel the hole)

$(x-1)$

$(5-1)$

$= 4$ $\quad \lim_{x \to 5} \dfrac{x^2-6x+5}{x-5} = 4$

**20.** $\lim_{t \to 1} \dfrac{t^4 - 1}{t^3 - 1}$

start by using difference of perfect squares to factor the numerator

$(t^2+1)(t^2-1) \leftarrow$ factor again

$(t^2+1)(t-1)(t-1)$

then factor the denominator

$(t-1)(t^2+t+1)$ check the factor

$= t^3 + t^2 + t - t^2 - t - 1$

$= t^3 - 1$ (factored correctly)

then you will be left with

$$\frac{(t^2+1)(t-1)(t+1)}{(t-1)(t^2+t+1)}$$

$(t-1)$ is on both the top and bottom of the fraction

$$\frac{(t^2+1)(t+1)}{t^2+t+1}$$

plug in 1 for $t$ and solve

$$\frac{(1^2+1)(1+1)}{1^2+1+1}$$

$$= \frac{(2)(2)}{3}$$

$$= \frac{4}{3}$$

**30.** $\lim\limits_{x \to -4} \dfrac{\sqrt{x^2+9} - 5}{x+4}$

Solution

start by multiplying the numerator and denominator by the conjugate of $\sqrt{x^2+9} - 5$

$$\lim_{x \to -4} \frac{\sqrt{x^2+9} - 5}{x+4} \left( \frac{\sqrt{x^2+9} + 5}{\sqrt{x^2+9} + 5} \right)$$

$= \lim \quad \underline{x^2+9-25}$ Simplify

$$x \to -4 \quad (x+4)(\sqrt{x^2+9}+5)$$

$$= \lim_{x \to -4} \frac{x^2-16}{(x+4)(\sqrt{x^2+9}+5)}$$ factor the numerator (difference of perfect squares)

$$= \lim_{x \to -4} \frac{(x+4)(x-4)}{(x+4)(\sqrt{x^2+9}+5)}$$ simplify

$$= \lim_{x \to -4} \frac{x-4}{\sqrt{x^2+9}+5}$$ plug in -4 for $x$

$$= \frac{-4-4}{\sqrt{-4^2+9}+5}$$ simplify

$$= \frac{-8}{\sqrt{25}+5}$$

$$= \frac{-8}{5+5}$$

$$= \frac{-8}{10}$$

$$= -\frac{4}{5}$$

## 2.4 The Precise Definition of a Limit

## Symbol/Abbreviation Review

$\forall$ → for all/for every

$\exists$ → There exists/there is

$\ni$ → such that (s.t.)

$\therefore$ → Therefore

$\varepsilon$ → epsilon (represents desired margin of error)

$\delta$ → delta (maximum distance from $x=a$ to fit in the margin of error)

W.T.S. → Want to show

How do you translate a limit statement to the $\varepsilon/\delta$ form?

$\lim_{x \to a} f(x) = L$ → if $0 < |x-a| < \delta$ then $|f(x) - L| < \varepsilon$

What is the general $\varepsilon/\delta$ Proof Statement?

W.T.S. $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. if $0 < |x-a| < \delta$, then $|f(x) - L| < \varepsilon$

What is the first step in solving a proof?

find $\delta$

What is the second step of the proof?

Plug in $\delta$ for $\varepsilon$ (you're undoing what you did in step 1)

What is the final step?

state the conclusion ($\therefore$ restate lim equation)

**16.** $\lim_{x \to 4} (2x - 5) = 3$

Solution

$0 < |x-4| < \delta \qquad |(2x-5)-3| < \varepsilon$

W.T.S. $\forall \varepsilon > 0, \exists \delta > 0$ s.t. if $0 < |x-4| < \delta$, then $|(2x-5)-3| < \varepsilon$

1) find $\delta$

$|(2x-5)-3| < \varepsilon$

$= |2x-8| < \varepsilon$

$= |2(x-4)| < \varepsilon$

$= 2|x-4| < \varepsilon$

$= |x-4| < \frac{\varepsilon}{2} \quad \to |x-4| < \delta$

$\frac{\varepsilon}{2} = \delta$

2) Prove it

Given $\varepsilon > 0$, let $\delta = \frac{\varepsilon}{2}$

$|x-4| < \frac{\varepsilon}{2}$

$= 2|x-4| < \varepsilon$

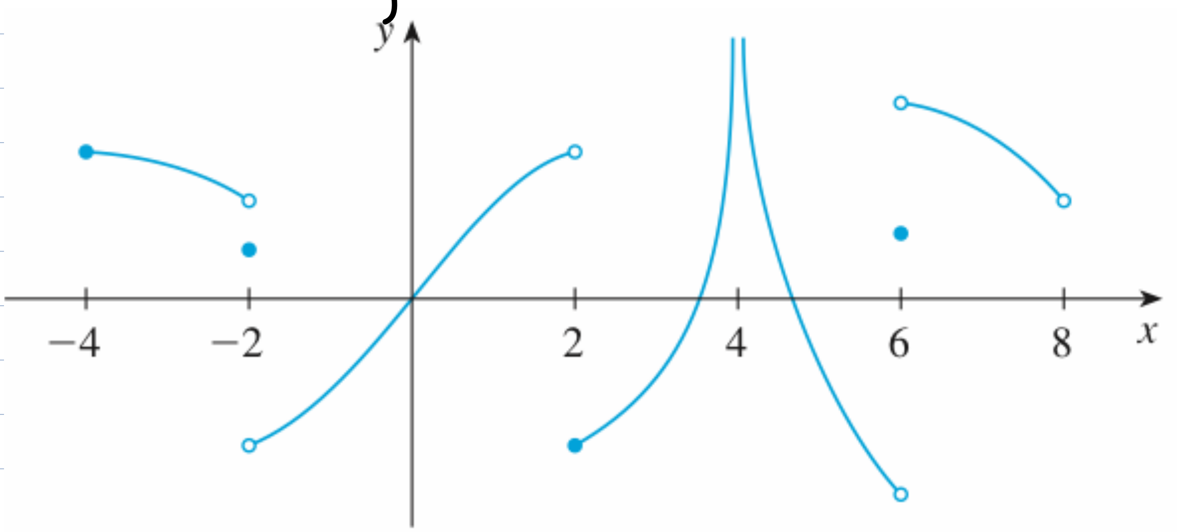
$= |2(x-4)| < \varepsilon$

$= |2x-8| < \varepsilon$

$= |(2x-5)-3| < \varepsilon$

$\therefore \lim_{x \to 4} (2x-5) = 3$

## 2.5 Continuity



$\lim_{x \to -2^-} f(x) =$

$\lim_{x \to -2^+} f(x) =$

$\lim_{x \to -2} f(x) =$

$f(-2) =$

$\lim_{x \to 0^-} f(x) =$

$\lim_{x \to 0^+} f(x) =$

$\lim_{x \to 0} f(x) =$

$f(0) =$

$\lim_{x \to 2^-} f(x) =$

$\lim_{x \to 2^+} f(x) =$

$\lim_{x \to 2} f(x) =$

$f(2) =$

$\lim_{x \to 4^-} f(x) =$

$\lim_{x \to 4^+} f(x) =$

$\lim_{x \to 4} f(x) =$

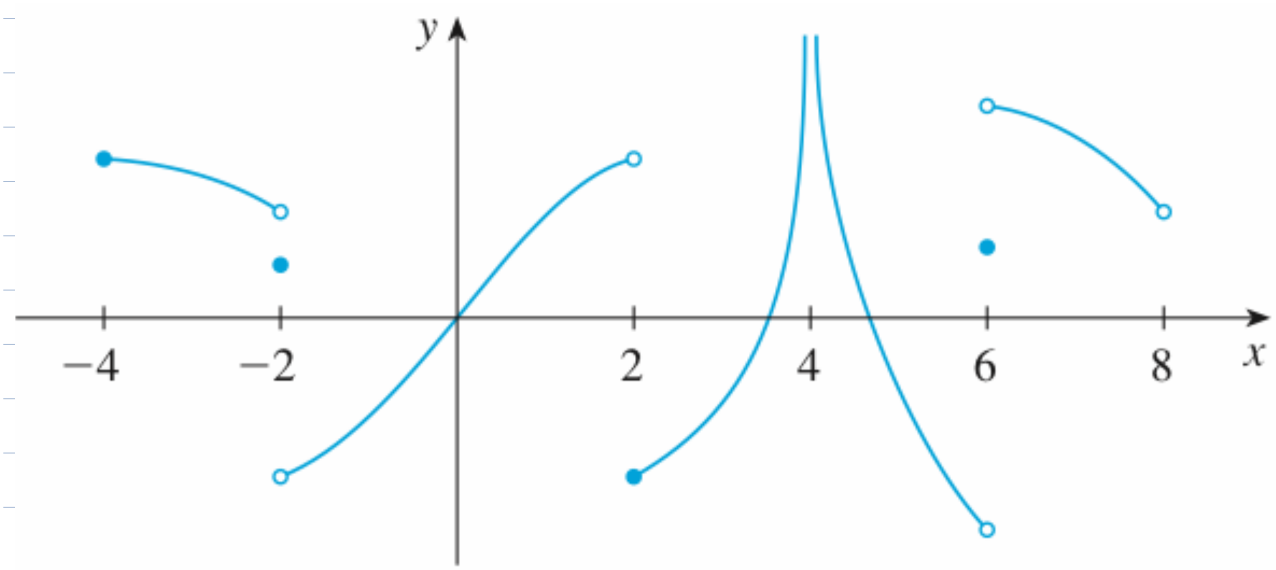
$f(4) =$

$\lim_{x \to 6^-} f(x) =$

$\lim_{x \to 6^+} f(x) =$

$\lim_{x \to 6} f(x) =$

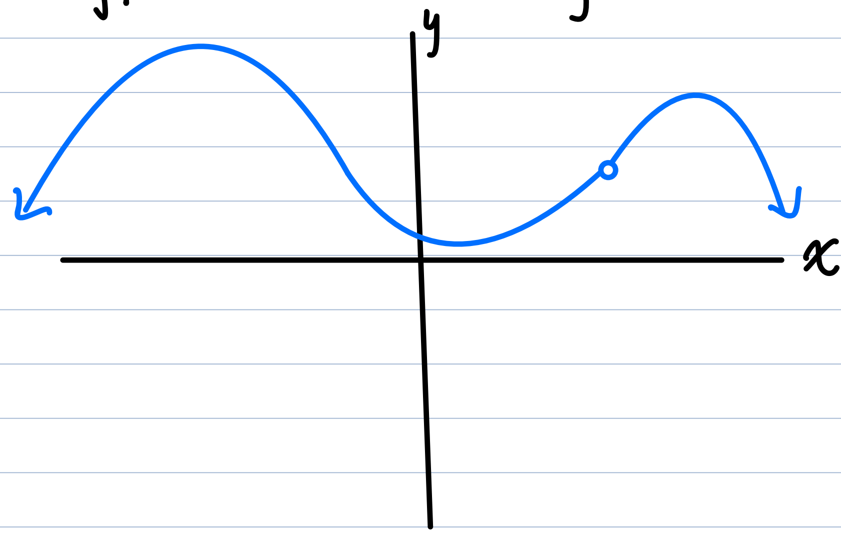
$f(6) =$



Is $f(x)$ continuous at the following points?
If not, what type of discontinuity is present?

$f(-2)$ $f(0)$ $f(2)$ $f(4)$ $f(6)$

What type of discontinuity is this?




Removable Point Discontinuity